Deriving the Sum equations:

The formulas below are used for computing the sums of powers of positive integers. The value of using formulas for computing these sums becomes apparent when we have a large number of terms to add. For instance, suppose you wanted to add the cubes of the first 100 whole numbers. Performing this number of additions and cubing on your calculator can be a daunting task, not to mention error prone. Once we have a formula for this sum then those types of calculations can be done quickly and accurately.

You will be deriving formulas for the sum of the first n positive integers, the sum of the squares of the first *n* positive integers, and the sum of the cubes of the first *n* positive integers.

There are the formulas that you will be proving:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Sum of the First *n* Positive Integers

Hint: The trick for solving this equation is a rearrangement of the sum as follows:

$S_n = 1 + 2 + 3 + \ldots + n - 1 + n$

Sum of the Squares of the First n Positive Integers

Hint 1: When I sum up 1+1+1+…1 n times I get *n*. So the sum of the 0th powers is linear in *n*.

When I sum up 1+2+…+n, I get *n*(*n*+1)/2, so the sum of first powers is quadratic in n.

Consider that the sum of squares is cubic in *n.*

Hint 2: (And it’s a big one)

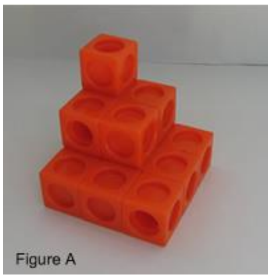


Figure A

If we wanted the sum of the first three squared integers, we could visualise these as blocks to make a single block in Figure A.

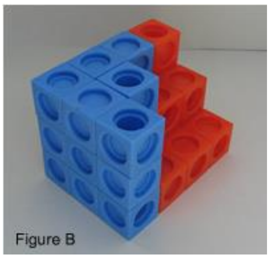


Figure B

Now treat the object in A as a single building block. If you put two of these building blocks together you get the solid in Figure B.

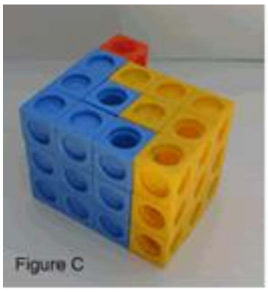


Figure C

Adding another building block you get the solid on the left in Figure C.

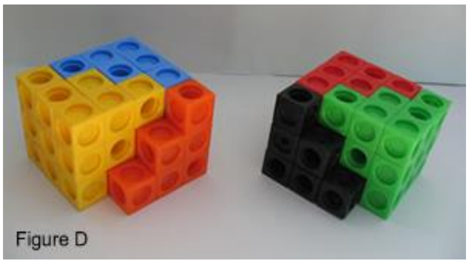


Figure D

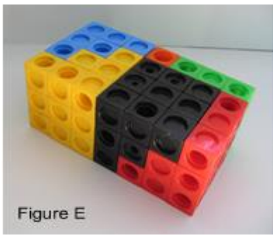


Figure E

Figures D and E show two copies of the solid made from three building blocks separately (D) and then placed together (E).

Well the task is to work out how many cubes are inside our original building block (A), and we can do this two ways.

By direct counting we get: 1 + 4 + 9 = 14.

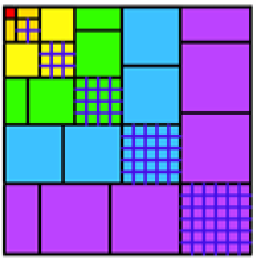
But we also have shown that six of our building blocks can be arranged into the solid cuboid in Figure E. So, how many cubes are there in Figure E? (Make sure you write down your working, it will be important for later)

Now here's a question:

Would the construction have worked if our building block had more layers, e.g. 1 cube on top of 4 cubes on top of 9 cubes on top of 16 cubes? Now how many cubes would be in your figure E?

Have you come up with a formula yet?

Sum of the Cubes of the First n Positive Integers



Have a look at the diagram to the left. It shows both those numbers squared, and those numbers cubed. Can you write an equation that shows a link between the two, and use this to come up with a formula to find the sum of the cubes of the first n positive integers?